

# New Implementation Techniques for the Exterior Stokes Problem in the Plane

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We present a method to solve numerically two-dimensional Stokes problems on exterior domains. Our scheme is based on the fully discrete BEM–FEM formulation proposed in [21] whose main advantage is that only elemental quadratures are used to approximate the weakly singular boundary integrals. We show in this article that it is possible to maintain this important property without using curved triangles in the discretization process. This modification makes the method easier to implement and the numerical experiments reveal that it still keeps the optimal order of convergence of the original scheme.

We also introduce in this paper a new iterative method to solve the complicated linear systems of equations that arises from this type of BEM–FEM discretizations. © 2001 Academic Press

*Key Words:* exterior Stokes problems; singular integral equations; finite element methods; boundary element methods.

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## 1. INTRODUCTION

The finite element method (FEM) and the boundary element method (BEM) are well-known procedures to approximate solutions of partial differential equations. On the one hand, the boundary element method is appropriate to solve problems in unbounded domains with the restriction that the equation should be linear, homogeneous, and with constant coefficients. On the other hand, the finite element method only works on bounded domains but it may be used for nonlinear and nonhomogeneous equations. Therefore, the advantages of each method compensate the deficiencies of the other one. Often, it is necessary to combine both of them to solve problems in exterior domains.

Much progress has been made in the numerical analysis of these methods since the first BEM–FEM coupling was introduced at the beginning of the 1980s; cf. [5, 11, 12, 17, 20, 21, 22]. However, a lot remains to be done before these coupling procedures become popular

tools for engineering calculations. For example, little is known about efficient algorithms to solve the complicated linear systems that arise from these formulations. Furthermore, the matrix assembly process requires the computation of integrals with nearly singular integrands over the auxiliary boundary. The design of efficient algorithms for this task is of great importance in order to improve the practicability of the method. Here, we will show how to handle these problems in the case of an exterior Stokes system.

The first BEM–FEM procedure for this problem was introduced by Sequeira in [22]. The formulation of Sequeira relies on the so-called *one boundary integral approach* introduced by Johnson and Nedelec for the Laplace equation [17]. The general Johnson–Nedelec procedure consists of dividing the unbounded domain into two subregions, a bounded inner region and an unbounded outer one, by introducing an auxiliary common boundary. This division is done so that the support of the right-hand side of the equation (i.e., the external forces) falls into the inner domain. An adapted Green formula, which makes use of the fundamental solution of the Stokes problem, gives an integral representation of the solution in the exterior domain. Next, this representation is used to deduce a nonlocal condition on the auxiliary boundary for the problem in the inner region. We point out that it is important to choose a smooth artificial interface in order to ensure the compactness of the double-layer potential which is essential for the analysis of the discrete problem, cf. [17, 20–22].

Usually, the discrete problem is posed on polygonal approximations of the auxiliary boundary. This strategy has a serious drawback since it renders difficult the approximation of the nearly singular boundary integrals by simple quadratures; see [17, 22, 23]. A more efficient method has been recently proposed in [21], where the integral operators are discretized on their natural boundary (the regular auxiliary interface); see also [20]. These authors are able to design a fully discrete formulation for the exterior Stokes system that requires few kernel evaluations while preserving the stability and convergence properties which are obtained when the integrals are computed exactly. This discretization method relies on exact triangulations of the domain. Hence, curved triangles are needed all along the auxiliary interface.

The aim of this paper is to show that actually one can find a compromise between the two previous discretization methods. We use straight triangles for the finite element part and discretize the boundary integral operators on the (regular) auxiliary boundary. The numerical experiments obtained in this paper show that this simplification does not affect the convergence properties proven in [21] for the original method.

The second objective of the paper concerns the algorithm proposed to solve the rather complicated linear systems of equations that arise from our BEM–FEM formulation of the Stokes problem. We use an iterative method that allows us to uncouple the boundary and the finite element methods. This means that at each iteration step we have to solve sequentially a usual Stokes problem by finite elements and a boundary integral equation. The advantage of this method is that we do not need to store the huge, unstructured, and nonsymmetric global matrix (see (19)) and the problems we have to solve during the iteration process are standard. Furthermore, we see from the numerical experiments described in this paper that the method is stable in the sense that the number of iterations does not increase with the number of unknowns.

The paper is organized as follows: In Sections 2 and 3 we introduce the model problem and formulate its Galerkin discretization. We provide the corresponding fully discrete scheme in Section 4. In Section 5 we present an iterative method to solve the system of linear equation. Finally, we give our numerical results in Section 6 and the conclusions in Section 7.

In the sequel, small boldface letters (capital boldface resp.) will denote vectors or vector valued functions (matrices or matrix valued functions resp.). Vectors in  $\mathbf{R}^2$  are always to be understood as column vectors, and subscripts will index their different components. The superscript  $\top$  will denote transposition of a matrix and a dot will denote the Euclidean inner product in  $\mathbf{R}^2$ , i.e.,

$$\mathbf{u} \cdot \mathbf{v} := \mathbf{u}^\top \mathbf{v} = \sum_{i=1}^2 u_i v_i.$$

Let  $\mathcal{O}$  be an open set or the boundary of a bounded domain in the plane. We denote in the following by  $(\cdot, \cdot)_{0,\mathcal{O}}$  the inner product in  $L^2(\mathcal{O})$ . Since we will deal with vector unknowns, we need product forms of some spaces. If  $H$  is a function space, we will denote  $\mathbf{H} := H \times H$  endowed with the product norm and corresponding inner product (when this exists). We will use the same notation for the inner product, since it will be clear from the context and notations used for functions, when scalar or vector functions are used.

## 2. STATEMENT OF THE PROBLEM

Let  $\Omega$  be a bounded domain in  $\mathbf{R}^2$  with Lipschitz boundary  $\Gamma$ , and let  $\Omega'$  be its exterior, i.e., the complement of its closure in  $\mathbf{R}^2$ . The steady-state exterior Stokes problem consists of finding a velocity field  $\mathbf{u}$  and a pressure field  $p$ , defined on  $\Omega'$ , satisfying

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega', \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega', \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma, \\ \mathbf{u} \text{ and } p &\text{ bounded as } |\mathbf{x}| \rightarrow \infty. \end{aligned} \tag{1}$$

We assume that the support of the external force function  $\mathbf{f}$  is bounded. We have also assumed that the dynamic viscosity equals 1.

Let  $\Omega_0$  be a simply connected bounded domain containing both  $\bar{\Omega}$  and the support of  $\mathbf{f}$  and such that its boundary  $\Gamma_0$  can be parameterized by a smooth function. Then  $\Gamma_0$  splits  $\Omega'$  into two subdomains,  $\Omega^- := \Omega_0 \cap \Omega'$  and  $\Omega^+ := \Omega'_0$ . Limits on  $\Gamma_0$  of functions defined on  $\Omega^+$  or  $\Omega^-$ , either in a classical or a weak sense, will be denoted simply by the superscript  $+$  or  $-$ , respectively.

For sufficiently smooth couples of velocity and pressure fields, we can define the outer and inner stress vector at the boundary  $\Gamma_0$  by the expressions

$$\mathbf{t}^\pm[\mathbf{u}, p] := -p^\pm \mathbf{n} + 2\mathbf{E}[\mathbf{u}]^\pm \mathbf{n},$$

where  $\mathbf{n}$  is the unit normal vector in  $\Gamma_0$  oriented from  $\Omega^-$  to  $\Omega^+$  (cf. Fig. 1) and  $\mathbf{E}[\mathbf{u}]$  is the velocity deformation tensor:

$$\mathbf{E}_{i,j}[\mathbf{u}] := \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (i, j = 1, 2).$$

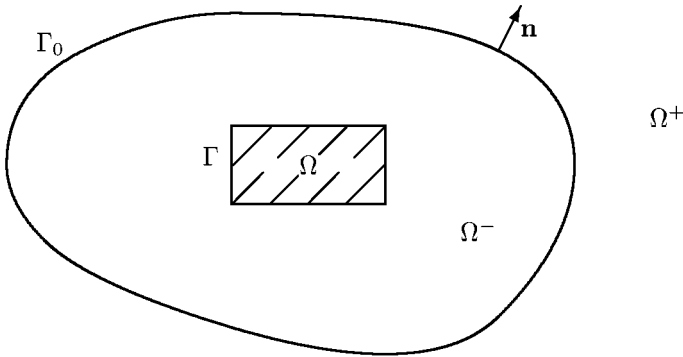


FIG. 1. Geometry of the problem.

Then we can write (1) as a Stokes problem in the bounded domain  $\Omega^-$ ,

$$\begin{aligned} -\Delta \mathbf{u}^- + \nabla p^- &= \mathbf{f} && \text{in } \Omega^-, \\ \nabla \cdot \mathbf{u}^- &= 0 && \text{in } \Omega^-, \\ \mathbf{u}^- &= \mathbf{0} && \text{on } \Gamma, \end{aligned} \quad (2)$$

coupled with the exterior and homogeneous Stokes problem,

$$\begin{aligned} -\Delta \mathbf{u}^+ + \nabla p^+ &= \mathbf{0} && \text{in } \Omega^+, \\ \nabla \cdot \mathbf{u}^+ &= 0, && \text{in } \Omega^+, \\ \mathbf{u} \text{ and } p &\text{ bounded as } |\mathbf{x}| \rightarrow \infty, \end{aligned} \quad (3)$$

by means of the following transmission conditions on  $\Gamma_0$

$$\begin{aligned} \mathbf{u}^- &= \mathbf{u}^+, \\ \mathbf{t}^-[\mathbf{u}, p] &= \mathbf{t}^+[\mathbf{u}, p]. \end{aligned} \quad (4)$$

The variational formulation of the interior problem follows from completely standard arguments [13]:

find  $\mathbf{u}^-$ ,  $p^-$ , such that

$$\begin{aligned} a(\mathbf{u}^-, \mathbf{v}) - (p^-, \nabla \cdot \mathbf{v})_{0, \Omega^-} &= (\mathbf{f}, \mathbf{v})_{0, \Omega^-} + (\mathbf{t}^-[\mathbf{u}, p], \mathbf{v})_{0, \Gamma_0} && \forall \mathbf{v} \\ (q, \nabla \cdot \mathbf{u}^-)_{0, \Omega^-} &= 0, && \forall q, \end{aligned} \quad (5)$$

with

$$a(\mathbf{u}^-, \mathbf{v}) := 2 \sum_{i,j=1}^2 \int_{\Omega^-} \mathbf{E}_{i,j}[\mathbf{u}^-] \mathbf{E}_{i,j}[\mathbf{v}] \, dx.$$

Note that we used here the identity

$$\Delta u_i^- = 2 \sum_{j=1}^2 \frac{\partial \mathbf{E}_{i,j}[\mathbf{u}^-]}{\partial x_j},$$

which is valid when  $\nabla \cdot \mathbf{u}^- = 0$ .

For the exterior problem we consider the classical formulation on the boundary of the exterior Stokes problems (see [22] or [11]). Let  $(\mathbf{U}, \mathbf{p})$  be the fundamental solution of the Stokes operator

$$\mathbf{p}(\mathbf{x} - \mathbf{y}) := \frac{1}{2\pi} \frac{1}{|\mathbf{x} - \mathbf{y}|^2} (\mathbf{x} - \mathbf{y}), \tag{6}$$

$$\mathbf{U}(\mathbf{x} - \mathbf{y}) := -\frac{1}{4\pi} \log |\mathbf{x} - \mathbf{y}| \mathbf{I} + \frac{1}{4\pi} \frac{1}{|\mathbf{x} - \mathbf{y}|^2} (\mathbf{x} - \mathbf{y})(\mathbf{x} - \mathbf{y})^\top, \tag{7}$$

where  $\mathbf{I}$  is the  $2 \times 2$  identity matrix. Denote

$$\mathbf{T}^\pm[\mathbf{U}, \mathbf{p}] := (\mathbf{t}^\pm[\mathbf{u}_1, p_1], \mathbf{t}^\pm[\mathbf{u}_2, p_2])^\top,$$

where  $\mathbf{u}_i$  are the column vectors of  $\mathbf{U}$ . Then the solution of the problem in  $\Omega^+$  can be represented from the boundary  $\Gamma_0$  as follows, cf. Theorem 3.2 in [10].

**THEOREM 2.1.** *There exist vector and scalar constants  $\mathbf{u}_0$  and  $p_0$  such that for almost all  $\mathbf{x} \in \Omega^+$*

$$\mathbf{u} = \mathbf{u}_0 + \int_{\Gamma_0} \mathbf{T}_y^+[\mathbf{U}(\cdot - \mathbf{y}), \mathbf{p}(\cdot - \mathbf{y})] \mathbf{u}^+(\mathbf{y}) \, d\sigma_y - \int_{\Gamma_0} \mathbf{U}(\cdot - \mathbf{y}) \mathbf{t}^+[\mathbf{u}, p](\mathbf{y}) \, d\sigma_y, \tag{8}$$

and

$$p = p_0 + 2 \int_{\Gamma_0} \frac{\partial}{\partial \mathbf{n}_y} \mathbf{p}(\cdot - \mathbf{y}) \cdot \mathbf{u}^+(\mathbf{y}) \, d\sigma_y - \int_{\Gamma_0} \mathbf{p}(\cdot - \mathbf{y}) \cdot \mathbf{t}^+[\mathbf{u}, p](\mathbf{y}) \, d\sigma_y. \tag{9}$$

Moreover, as  $|\mathbf{x}| \rightarrow \infty$

$$\mathbf{u}^+(\mathbf{x}) = \mathbf{U}(\mathbf{x})\delta + \mathbf{u}_0 + O\left(\frac{1}{|\mathbf{x}|}\right)$$

and

$$p^+(\mathbf{x}) = p_0 + \mathbf{p}(\mathbf{x}) \cdot \delta + O\left(\frac{1}{|\mathbf{x}|^2}\right)$$

where  $\delta := \int_{\Gamma_0} \mathbf{t}^+[\mathbf{u}, p] \, d\sigma$ .

The subscript  $\mathbf{y}$  in the operator  $\mathbf{T}^+$  in (8) and in the normal derivative in (9) denotes differentiation with respect to the  $\mathbf{y}$  variables. In (8) integration is to be understood componentwise.

Note that, as  $\mathbf{U}(\mathbf{x}) = O(\log(\mathbf{x}))$  when  $|\mathbf{x}| \rightarrow \infty$ , the asymptotic behavior imposed in (3) is satisfied if and only if  $\delta = \mathbf{0}$ . We also observe that Theorem 2.1 ensures the existence of a well-defined vector  $\mathbf{u}_0$  to which  $\mathbf{u}$  tends at large distances. However, in dimension two,  $\mathbf{u}_0$  cannot be prescribed a priori. This phenomenon, related only to the bidimensional problem, is known as the Stokes paradox, cf. [10].

The usual way of dealing with (8) is taking the limit from  $\Omega^+$  to  $\Gamma_0$  and obtaining an integral identity relating  $\mathbf{u}^+$  and  $\mathbf{t}^+[\mathbf{u}, p]$  to each other:

$$\begin{aligned} \frac{1}{2}\mathbf{u}^+ - \int_{\Gamma_0} \mathbf{T}_y^+[U(\cdot - \mathbf{y}), \mathbf{p}(\cdot - \mathbf{y})]\mathbf{u}^+(\mathbf{y}) \, d\sigma_y \\ + \int_{\Gamma_0} U(\cdot - \mathbf{y})\mathbf{t}^+[\mathbf{u}, p](\mathbf{y}) \, d\sigma_y - \mathbf{u}_0 = \mathbf{0}, \quad \text{on } \Gamma_0. \end{aligned} \tag{10}$$

This identity is usually used (see [22]) as a nonlocal boundary condition on  $\Gamma_0$  for problem (5) to obtain the solution in  $\overline{\Omega^-}$ , and Eqs. (8) and (9) give representation formulas for the solution in  $\Omega^+$ .

It is worthwhile noting that identity

$$\int_{\Gamma_0} U(\mathbf{x} - \mathbf{y})\mathbf{n}(\mathbf{y}) \, d\sigma_y = \mathbf{0}, \quad \text{on } \Gamma_0 \tag{11}$$

is obtained by just writing (10) for the particular solution of the Stokes problem which consists of a null velocity field and a constant pressure. Equation (11) is useful to show that the pressure  $p^-$  is only determined up to an additive constant by (5) and (10) since these equations remain invariant when we substitute  $p^-$  by  $p^- + c$  (which implies that we have to substitute  $\mathbf{t}^\pm[\mathbf{u}, p]$  by  $\mathbf{t}^\pm[\mathbf{u}, p + c] = -c\mathbf{n} + \mathbf{t}^\pm[\mathbf{u}, p]$ ). Hence, we must add a condition to fix the pressure. This may be performed in several ways but it is convenient to impose the restriction

$$\int_{\Gamma_0} \mathbf{t}^+[\mathbf{u}, p] \cdot \mathbf{n} \, d\sigma = 0 \tag{12}$$

in order to determine the pressure without perturbing the stress solution vector.

Instead of using (10) directly as in [22], we follow [21] and parameterize the auxiliary boundary  $\Gamma_0$ . Then, we change all functions (resp. equations) defined on this boundary for the corresponding periodic functions (resp. equations).

Let  $\mathbf{x} : \mathbf{R} \rightarrow \mathbf{R}^2$  be a smooth regular 1-periodic parametric representation of the curve  $\Gamma_0$ , such that

$$|\mathbf{x}'(s)| > 0, \quad \forall s \in \mathbf{R}, \quad \text{and} \quad \mathbf{x}(t) \neq \mathbf{x}(s), \quad \text{for } 0 < |t - s| < 1.$$

This parameterization of  $\Gamma_0$  allows us to define the inner parameterized trace onto  $\Gamma_0$  as the unique extension of

$$\begin{aligned} \gamma : \mathcal{C}^\infty(\overline{\Omega^-}) &\rightarrow L^2(0, 1) \\ u &\mapsto u|_{\Gamma_0}(\mathbf{x}(\cdot)) \end{aligned}$$

to the whole of  $H^1(\Omega^-)$ . Theorem 8.15 of [18] proves that  $\gamma : H^1(\Omega^-) \rightarrow H^{1/2}$  is bounded and onto, where  $H^{1/2}$  is the completion of the space of 1-periodic infinitely differentiable real valued functions with the norm

$$\|g\|_{1/2} := \left( \sum_{k \in \mathbf{Z}} (1 + |k|^2)^{1/2} |\hat{g}(k)|^2 \right)^{1/2}.$$

We will denote by  $H^{-1/2}$  the dual space of  $H^{1/2}$ . The  $L^2(0, 1)$ -inner product

$$(\lambda, \mu) := \int_0^1 \lambda(s)\mu(s) ds$$

can be extended to represent the duality of  $H^{-1/2}$  and  $H^{1/2}$ . We will keep the same notation for this duality bracket.

Consider the following integral operators:

$$Vg := \int_0^1 \mathbf{V}(\cdot, t)g(t) dt, \quad Kg := \int_0^1 \mathbf{K}(\cdot, t)g(t) dt, \tag{13}$$

where

$$\mathbf{V}(s, t) := \mathbf{U}(\mathbf{x}(s) - \mathbf{x}(t)) \tag{14}$$

and

$$\begin{aligned} \mathbf{K}(s, t) &:= |\mathbf{x}'(t)| \mathbf{T}_{\mathbf{x}(t)}^+ [\mathbf{U}(\mathbf{x}(s) - \mathbf{x}(t)), \mathbf{p}(\mathbf{x}(s) - \mathbf{x}(t))] \\ &= -\frac{|\mathbf{x}'(t)|}{\pi} \frac{(\mathbf{x}(s) - \mathbf{x}(t)) \cdot \mathbf{n}(t)}{|\mathbf{x}(s) - \mathbf{x}(t)|^4} (\mathbf{x}(s) - \mathbf{x}(t))(\mathbf{x}(s) - \mathbf{x}(t))^\top. \end{aligned} \tag{15}$$

These operators are parameterized versions of classical boundary integral operators used in [22].

Let us denote

$$\lambda(t) := |\mathbf{x}'(t)| \mathbf{t}^+ [\mathbf{u}, p](\mathbf{x}(t)).$$

It is straightforward from (10) that

$$V\lambda + \left(\frac{1}{2}I - K\right) \gamma \mathbf{u} - \mathbf{u}_0 = \mathbf{0},$$

where  $I$  is the identity operator. Notice that condition  $\delta = \mathbf{0}$  reads now

$$\int_0^1 \lambda(s) ds = \mathbf{0},$$

where the integrals are applied componentwise.

We are now in position to give the global weak formulation of (1),

$$\begin{aligned} \text{find } (\mathbf{u}, p, \lambda) &\in \mathbf{H}_\Gamma^1(\Omega^-) \times L^2(\Omega^-) \times \mathbf{H}_0^{-\frac{1}{2}}, \text{ such that} \\ a(\mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v})_{0, \Omega^-} - (\gamma \mathbf{v}, \lambda) &= (\mathbf{f}, \mathbf{v})_{0, \Omega^-}, \quad \forall \mathbf{v} \in \mathbf{H}_\Gamma^1(\Omega^-) \\ (q, \nabla \cdot \mathbf{u})_{0, \Omega^-} &= 0, \quad \forall q \in L^2(\Omega^-), \\ (2V\lambda, \mu) + (\gamma \mathbf{u}, \mu) - (2K\gamma \mathbf{u}, \mu) &= 0, \quad \forall \mu \in \mathbf{H}_0^{-\frac{1}{2}} \\ \int_0^1 \lambda(t) \cdot \mathbf{n}(\mathbf{x}(t)) dt &= 0, \end{aligned} \tag{16}$$

where the last equation is restriction (12) given in terms of  $\lambda$ . The zero mean value condition for each component of  $\lambda$  is included in the space

$$\mathbf{H}_0^{-\frac{1}{2}} := \{\mu \in \mathbf{H}^{-\frac{1}{2}} : (\mu, \mathbf{c}) = 0 \quad \forall \mathbf{c} \in \mathbf{R}^2\}.$$

Finally, we denoted

$$\mathbf{H}_\Gamma^1(\Omega^-) := \{v \in \mathbf{H}^1(\Omega^-) : v|_\Gamma = \mathbf{0}\}.$$

### 3. A BEM–FEM DISCRETIZATION

For simplicity in the exposition, we restrict ourselves to polygonal boundaries  $\Gamma$ . Given  $h := 1/N$ , with  $N$  a positive integer, let  $t_i := ih$  be the induced uniform partition of  $\mathbf{R}$ . We denote by  $\Omega_h$  the polygonal domain delimited by the polygonal boundary  $\Gamma_{0,h}$  whose vertices are  $\{\mathbf{x}(t_i) : i = 1, \dots, N\}$  and  $\Gamma$ . Let  $\tau_h$  be a regular triangulation of  $\bar{\Omega}_h$  formed by triangles such that: (a) there exists a constant  $C > 0$  such that for all  $T \in \tau_h$ ,  $h_T \leq Ch$  (where  $h_T$  is the diameter of  $T$ ); (b) any vertex of a triangle lying on the exterior boundary  $\Gamma_{0,h}$  of  $\partial\Omega_h$  belongs to  $\{\mathbf{x}(t_i) : i = 1, \dots, N\}$ .

Let  $T$  be an arbitrary triangle of  $\tau_h$  with vertices  $\mathbf{a}_1^T$ ,  $\mathbf{a}_2^T$ , and  $\mathbf{a}_3^T$ . We denote by  $f_i^T$  the side of  $T$  opposite to  $\mathbf{a}_i^T$  and by  $\mathbf{n}_{i,T}$  the unit outward normal to  $f_i^T$ . We define the space

$$P_1(T) := \{p : T \rightarrow \mathbf{R} : p \in P_1\},$$

where  $P_1$  is the space of polynomials of degree not greater than one. We recall that the barycentric coordinate functions  $\lambda_{i,T} \in P_1(T)$  ( $i = 1, 2, 3$ ) are uniquely determined by  $\lambda_{i,T}(\mathbf{a}_j^T) = \delta_{i,j}$ . For  $1 \leq i \leq 3$ , we introduce the functions

$$\mathbf{q}_{i,T} := \lambda_{j,T} \lambda_{k,T} \mathbf{n}_{i,T}, \quad (i, j, k) \in \mathcal{C}_3 := \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}.$$

It is well-known that the discrete approximation spaces for the pressure and the velocity cannot be chosen independently. They must satisfy the well known inf–sup condition. Among the numerous stable finite elements for the Stokes problem (cf. [2, 13]) we decided to use (as in [21]) the one introduced by Bernardi and Raugel in [1] because it requires few degrees of freedom. Indeed, the local discrete velocities space is given by

$$\mathbf{P}(T) := P_1(T) \oplus \text{span}(\mathbf{q}_{1,T}, \mathbf{q}_{2,T}, \mathbf{q}_{3,T}),$$

and it is easy to see that a function  $\phi \in \mathbf{P}(T)$  is uniquely determined by the nine degrees of freedom given by the Lagrange functionals

$$N_{i,T}(\phi) := \phi(\mathbf{a}_i^T), \quad (i = 1, 2, 3)$$

plus the moments

$$m_{i,T}(\phi) := \int_{f_i^T} \phi \cdot \mathbf{n}_{i,T} \, d\sigma \quad (i = 1, 2, 3).$$



Moreover, if  $(i, j, k) \in \mathcal{C}_3, \phi \in \mathbf{P}(T)$  and  $\phi(\mathbf{a}_i^T) = \phi(\mathbf{a}_j^T) = \mathbf{0}, m_{k,T}(\phi) = 0$ , then  $\phi|_{f_k^T} \equiv \mathbf{0}$ . Hence, we may define the global finite element space for the velocity by

$$\mathbf{W}_h := \{ \mathbf{v} \in \mathcal{C}^0(\Omega_h, \mathbf{R}^2) : \mathbf{v}|_T \in \mathbf{P}(T), \forall T \in \tau_h \} \cap \mathbf{H}_\Gamma^1(\Omega_h).$$

The corresponding space for the pressure is given by piecewise constant functions:

$$\mathcal{Q}_h := \{ q \in L^2(\Omega_h) : q|_T \in P_0, \forall T \in \tau_h \}.$$

Finally, we use splines of degree one for the unknown in the boundary

$$\mathbf{H}_h := \{ \boldsymbol{\mu} \in \mathbf{L}^2(0, 1) : \boldsymbol{\mu}|_{(t_i, t_{i+1})} \in P_0 \times P_0 (i = 1, \dots, N), \int_0^1 \boldsymbol{\mu} = \mathbf{0} \}.$$

Now, we need to define a discrete counterpart  $\gamma_h$  of the parameterized trace operator  $\gamma$ . This discrete linear operator will relate the space of traces  $\mathbf{W}_h(\Gamma_{0,h}) := \{ \mathbf{v}|_{\Gamma_{0,h}} \mathbf{v} \in \mathbf{W}_h \}$  of functions in  $\mathbf{W}_h$  to the subspace  $\mathbf{T}_h \subset \mathbf{H}^{1/2}$  defined by the set of vectorial functions whose components are continuous, 1-periodic and piecewise linear. It is clear that

$$\begin{aligned} \gamma_h : \mathbf{W}_h(\Gamma_{0,h}) &\rightarrow \mathbf{T}_h \\ \mathbf{v}|_{\Gamma_{0,h}} &\mapsto \gamma_h \mathbf{v} \end{aligned}$$

is uniquely determined by the conditions  $\gamma_h \mathbf{v}(t_i) := \mathbf{v}(\mathbf{x}(t_i))$  for  $i = 1, \dots, N$ . This operator allows us to ensure compatibility between the finite and boundary element meshes.

Note that  $\gamma_h$  transports only the information given by the linear part of the functions in  $\mathbf{W}_h(\Gamma_{0,h})$  and ignores the quadratic components. This is not expected to affect the convergence properties proven in [21] since the role of the quadratic parts of functions in  $\mathbf{W}_h$  is limited to ensure the inf–sup condition between this space and  $\mathcal{Q}_h$ . The numerical experiments of Section 6 confirm that there is no loss of accuracy.

We are now in position to write the discrete problem associated to (16),

$$\begin{aligned} \text{find } (\mathbf{u}_h, p_h, \boldsymbol{\lambda}_h) \in \mathbf{W}_h \times \mathcal{Q}_h \times \mathbf{H}_h, \text{ such that} \\ a_h(\mathbf{u}_h, \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v})_{0, \Omega_h} - (\gamma_h \mathbf{v}, \boldsymbol{\lambda}_h) &= (\mathbf{f}, \mathbf{v})_{0, \Omega_h}, \quad \forall \mathbf{v} \in \mathbf{W}_h \\ (q, \nabla \cdot \mathbf{u}_h)_{0, \Omega_h} &= 0, \quad \forall q \in \mathcal{Q}_h \\ (2V \boldsymbol{\lambda}_h, \boldsymbol{\mu}) + (\gamma_h \mathbf{u}_h, \boldsymbol{\mu}) - (2K \gamma_h \mathbf{u}_h, \boldsymbol{\mu}) &= 0, \quad \forall \boldsymbol{\mu} \in \mathbf{H}_h \end{aligned} \tag{17}$$

$$\int_0^1 \boldsymbol{\lambda}_h(t) \cdot \mathbf{n}(\mathbf{x}(t)) dt = 0.$$

The bilinear form  $a_h(\cdot, \cdot)$  is the restriction of  $a(\cdot, \cdot)$  to  $\Omega_h$ :

$$a_h(\mathbf{u}, \mathbf{v}) := 2 \sum_{i,j=1}^2 \int_{\Omega_h} \mathbf{E}_{i,j}[\mathbf{u}] \mathbf{E}_{i,j}[\mathbf{v}] dx.$$

#### 4. FULL DISCRETIZATION OF THE EQUATIONS

In this section we give a fully discrete scheme based on the application of numerical integration to the equations of the Galerkin method. We begin with the right-hand side of

the first equation of (17). Assuming that  $\mathbf{f}$  is continuous in  $\overline{\Omega^-}$ , we approximate for all  $\mathbf{v} \in \mathbf{W}_h$

$$(\mathbf{f}, \mathbf{v})_{0, \Omega_h} \simeq L_h(\mathbf{v}) := \sum_{T \in \tau_h} \frac{\text{mes}(T)}{3} \sum_{i=1}^3 (\mathbf{f} \cdot \mathbf{v})(\mathbf{a}_i^T),$$

where  $\mathbf{a}_i^T$  are the vertices of  $T$ .

Note that, in contrast to the method proposed in [21], here all the other integrals over  $\Omega_h$  can be computed exactly, since the integrands are polynomial functions. In practice one uses quadratures that give the exact value of the integrals. A formula of degree two on each triangle is sufficient for all cases.

The simplest boundary integral term can also be computed exactly by applying the midpoint quadrature formula,

$$(\gamma_h \mathbf{u}, \boldsymbol{\mu}) = h \sum_{i=1}^N \boldsymbol{\mu}_i \cdot \gamma_h \mathbf{u}(t_i + h/2), \quad \forall \mathbf{u} \in \mathbf{W}_h, \forall \boldsymbol{\mu} \in \mathbf{H}_h$$

where  $\boldsymbol{\mu}_i$  is the constant value of  $\boldsymbol{\mu}$  in  $(t_i, t_{i+1})$ .

We use the bidimensional midpoint quadrature formula to approximate the bilinear forms associated to both the single and double layer potentials. Indeed, we define

$$(K \gamma_h \mathbf{v}, \boldsymbol{\mu}) \simeq c_h(\gamma_h \mathbf{v}, \boldsymbol{\mu}) := h^2 \sum_{i,j=1}^N \boldsymbol{\mu}_i^\top \mathbf{K}(t_i + h/2, t_j + h/2) \gamma_h \mathbf{v}(t_j + h/2),$$

for all  $(\mathbf{v}, \boldsymbol{\mu}) \in \mathbf{W}_h \times \mathbf{H}_h$ . We point out here that we are using a formula that is one degree less than the one proposed in [21] for this term. This is because the polynomial function  $\gamma_h \mathbf{v}$  is also one degree less than the corresponding one in [21].

Numerical quadratures must be handled with care when defining an approximation of  $(V \boldsymbol{\lambda}, \boldsymbol{\mu})$  on  $\mathbf{H}_h \times \mathbf{H}_h$  because of the logarithmic singularity of the kernel  $V$ . Here, we consider the following decomposition of the kernel:

$$V(s, t) = -\frac{1}{4\pi} \log |s - t| \mathbf{I} + \mathbf{B}(s, t).$$

Notice that  $\mathbf{B}$  is of class  $\mathcal{C}^\infty$  in the domain  $\{(s, t) : |s - t| < 1\}$ . Now, the strategy consists in approximating the second integral and computing the first one exactly (cf. [16]); i.e.,

$$\int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} V(s, t) ds dt \simeq \mathbf{V}_{i,j} := -h^2 \frac{\log |h| + \kappa_{i-j}}{4\pi} \mathbf{I} + h^2 \mathbf{B}(t_i + h/2, t_j + h/2),$$

with

$$\kappa_k := \int_0^1 \int_0^1 \log |k + t - u| dt du$$

and

$$(\underline{i}, \underline{j}) := \begin{cases} (i, j), & \text{if } |i - j| \leq N/2, \\ (i, j - N), & \text{if } i - j > N/2, \\ (i - N, j), & \text{if } j - i > N/2. \end{cases}$$

Notice that the periodicity of  $V(\cdot, \cdot)$  allows one to use the indices  $(\underline{i}, \underline{j})$  instead of  $(i, j)$  and avoid the neighborhood of the region  $\{(s, t); |s - t| = 1\}$ . Then we approximate for all  $\lambda, \mu \in \mathbf{H}_h$

$$(V\lambda, \mu) \simeq d_h(\lambda, \mu) := \sum_{i,j=1}^N \mu_j^\top V_{i,j} \lambda_i.$$

Finally, we approximate the last equation of (17) by using again the one dimensional midpoint quadrature formula:

$$\int_0^1 \mu(t) \cdot \mathbf{n}(\mathbf{x}(t)) dt \simeq \ell_h(\mu) := h \sum_{i=1}^N \mu_i \cdot \mathbf{n}(\mathbf{x}(t_i + h/2)) \quad \forall \mu \in \mathbf{H}_h.$$

We are now in a position to write a fully discrete method for (16):

$$\begin{aligned} \text{find } (\bar{\mathbf{u}}_h, \bar{p}_h, \bar{\lambda}_h) \in \mathbf{W}_h \times \mathcal{Q}_h \times \mathbf{H}_h, \text{ such that} \\ a_h(\bar{\mathbf{u}}_h, \mathbf{v}) - (\bar{p}_h, \nabla \cdot \mathbf{v})_{0, \Omega_h} - (\gamma_h \mathbf{v}, \bar{\lambda}_h) = \mathbf{L}_h(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{W}_h \\ (q, \nabla \cdot \bar{\mathbf{u}}_h)_{0, \Omega_h} = 0, \quad \forall q \in \mathcal{Q}_h \\ 2d_h(\bar{\lambda}_h, \mu) + (\gamma_h \bar{\mathbf{u}}_h, \mu) - 2c_h(\gamma_h \bar{\mathbf{u}}_h, \mu) = 0, \quad \forall \mu \in \mathbf{H}_h \\ \ell_h(\bar{\lambda}_h) = 0. \end{aligned} \tag{18}$$

### 5. THE ITERATIVE METHOD

Let us denote by  $N_h, M_h$  and  $Z_h$  the dimensions of  $\mathbf{W}_h, \mathcal{Q}_h$  and  $\mathbf{H}_h$ , respectively, and by  $\{\varphi_i; i = 1, \dots, N_h\}, \{w_m; m = 1, \dots, M_h\}$  and  $\{\rho_l; l = 1, \dots, Z_h\}$  the basis for  $\mathbf{W}_h, \mathcal{Q}_h$  and  $\mathbf{H}_h$  respectively. If we set

$$\bar{\mathbf{u}}_h(\mathbf{x}) = \sum_{i=1}^{N_h} \mathbf{u}_i \varphi_i(\mathbf{x}), \quad \bar{p}_h(\mathbf{x}) = \sum_{m=1}^{M_h} p_m w_m(\mathbf{x}), \quad \bar{\lambda}_h(t) = \sum_{l=1}^{Z_h} \lambda_l \rho_l(t)$$

the linear system associated with (18) takes the following form

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^\top & \mathbf{C}^\top \\ \mathbf{B} & \mathbf{0} & \mathbf{0} \\ \mathbf{C} + \mathbf{K} & \mathbf{0} & -\mathbf{V} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \tag{19}$$

and the linear restriction on  $\bar{\lambda}_h$  may be written as

$$\mathbf{r}^\top \lambda = 0,$$

where

$$\begin{aligned} \mathbf{A}_{i,j} &:= a_h(\varphi_j, \varphi_i), & \mathbf{B}_{m,i} &:= -(w_m, \nabla \cdot \varphi_i)_{0,\Omega_h}, & \mathbf{C}_{i,l} &:= -(\gamma_h \varphi_i, \rho_l), \\ \mathbf{K}_{i,l} &:= 2c_h(\gamma_h \varphi_i, \rho_l), & \mathbf{V}_{l,k} &:= 2d_h(\rho_k, \rho_l), & f_i &:= L_h(\varphi_i) \end{aligned}$$

and, finally, the vector  $\mathbf{r}$  is defined by

$$\mathbf{r}_l := \ell_h(\rho_l).$$

The system (19) is not symmetric and badly structured since  $\mathbf{A}$ ,  $\mathbf{B}$ , and  $\mathbf{C}$  are sparse matrices while  $\mathbf{V}$  and  $\mathbf{K}$  are full. It is clear that the global matrix is too large to be stored and handled. The purpose of this section is to derive a convenient iterative method inspired from [9] to solve (18).

Starting with an initial guess  $(\mathbf{u}_h^0, \lambda_h^0) \in \mathbf{W}_h \times \mathbf{H}_h$ , for  $m \geq 1$ , we construct by induction the sequence  $(\mathbf{u}_h^m, p_h^m, \lambda_h^m) \in \mathbf{W}_h \times Q_h \times \mathbf{H}_h$  in the following steps:

1. Solve for  $(\mathbf{u}_h^m, p_h^m) \in \mathbf{W}_h \times Q_h$  the Stokes problem

$$\begin{aligned} a_h(\mathbf{u}_h^m, \mathbf{v}) - (p_h^m, \nabla \cdot \mathbf{v})_{0,\Omega_h} &= L_h(\mathbf{v}) + (\gamma_h \mathbf{v}, \lambda_h^{m-1}), & \forall \mathbf{v} \in \mathbf{W}_h, \\ (q, \nabla \cdot \mathbf{u}_h^m)_{0,\Omega_h} &= 0, & \forall q \in Q_h. \end{aligned} \tag{20}$$

2. Define

$$\mathbf{u}_h^m(\theta) := (1 - \theta)\mathbf{u}_h^m + \theta\mathbf{u}_h^{m-1},$$

where  $\theta \in [0, 1)$  is a relaxation parameter.

3. Solve for  $\lambda_h^m \in \mathbf{H}_h$  the integral equation

$$2d_h(\lambda_h^m, \mu) = -(\gamma_h \mathbf{u}_h^m(\theta), \mu) + 2c_h(\gamma_h \mathbf{u}_h^m(\theta), \mu), \quad \forall \mu \in \mathbf{H}_h, \tag{21}$$

subject to the linear restriction

$$\ell_h(\lambda_h^m) = 0. \tag{22}$$

The previous steps can be interpreted in matrix form as follows:

1. Solve for  $(\mathbf{u}^m, \mathbf{p}^m) \in \mathbf{R}^{N_h} \times \mathbf{R}^{M_h}$  the linear system

$$\begin{pmatrix} \mathbf{A} & \mathbf{B}^\top \\ \mathbf{B} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}^m \\ \mathbf{p}^m \end{pmatrix} = \begin{pmatrix} \mathbf{f} - \mathbf{C}^\top \lambda^{m-1} \\ \mathbf{0} \end{pmatrix}. \tag{23}$$

2. Set

$$\mathbf{u}^m(\theta) = (1 - \theta)\mathbf{u}^m + \theta\mathbf{u}^{m-1}, \quad (\theta \in [0, 1)).$$

3. Solve for  $(\alpha^m, \lambda^m) \in \mathbf{R}^{Z_h+1}$

$$\begin{pmatrix} \mathbf{0} & \mathbf{r}^\top \\ \mathbf{r} & \mathbf{V} \end{pmatrix} \begin{pmatrix} \alpha^m \\ \lambda^m \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -(\mathbf{C} + \mathbf{K})\mathbf{u}^m(\theta) \end{pmatrix}, \tag{24}$$

where restriction (22) has been incorporated into the linear system through the Lagrange multiplier  $\alpha^m$ , cf. [4].

There are a great number of techniques to solve the linear system of equations (23), cf. [8]. Here we simply use the well-known *pressure matrix method*. This consists of writing a linear system of equations for the pressure after elimination of the velocity vector. The resulting problem has a well-conditioned matrix  $\mathbf{BA}^{-1}\mathbf{B}^\top$  and hence, it may be solved efficiently by the conjugate gradient method. This entails the solution of a linear system with the same matrix  $\mathbf{A}$  at each iteration step of the conjugate gradient algorithm. Since  $\mathbf{A}$  is symmetric and positive definite, this can be performed by a direct method through a Cholesky decomposition of  $\mathbf{A}$ .

We also use a direct method to solve (24). Note that the matrix of the system is symmetric but indefinite (neither positive definite nor negative definite) and the Cholesky decomposition cannot be used. However, it turns out that it is still possible to factorize the matrix in half the work and space required for the standard Gaussian elimination. There exists a LAPACK subroutine for such a factorization, and the corresponding algorithm is described in [3].

Finally, we point out that we used a special storage of matrices  $\mathbf{A}$ ,  $\mathbf{B}$ ,  $\mathbf{C}$ , and  $\mathbf{K}$  that ignores the large zero blocks in order to increase the computational efficiency of the method.

## 6. NUMERICAL RESULTS

We test our numerical technique for a Stokes problem posed in the exterior of an ellipse centered at the origin with a major semiaxis of length 2 and a minor semiaxis of length 1. The analytical solution of the problem is given explicitly by

$$u_1(\mathbf{x}) = \frac{1}{4} \left[ \frac{x_1^2}{x_1^2 + (x_2 - 0.5)^2} - \frac{x_1^2}{x_1^2 + (x_2 + 0.5)^2} - \frac{1}{2} \log \frac{x_1^2 + (x_2 - 0.5)^2}{x_1^2 + (x_2 + 0.5)^2} \right]$$

$$u_2(\mathbf{x}) = \frac{1}{4} \left[ \frac{x_1(x_2 - 0.5)}{x_1^2 + (x_2 - 0.5)^2} - \frac{x_1(x_2 + 0.5)}{x_1^2 + (x_2 + 0.5)^2} \right]$$

and

$$p(\mathbf{x}) = \frac{1}{2} \left[ \frac{x_1}{x_1^2 + (x_2 - 0.5)^2} - \frac{x_1}{x_1^2 + (x_2 + 0.5)^2} \right].$$

A direct calculation shows that the external force  $\mathbf{f}$  corresponding to this solution vanishes identically. Instead, we have a nonhomogeneous Dirichlet condition on the ellipse  $\Gamma$  but this does not add any new difficulty to our problem. Let us consider the ellipse of minor and major semiaxis  $a = 2$  and  $b = 3$  as an auxiliary boundary  $\Gamma_0$ .

In all that follows, the iterative method is initialized with zero and iterations are continued until a reduction of  $10^{-6}$  is achieved in the relative residual  $\frac{|\lambda^m - \lambda^{m-1}|}{|\lambda^m|}$ .

We first give a qualitative comparison between the analytical solution and the numerical solution obtained on a mesh whose parameter is  $h = 1/64$  for the velocity field and  $h = 1/128$  for the pressure and the stress vector.

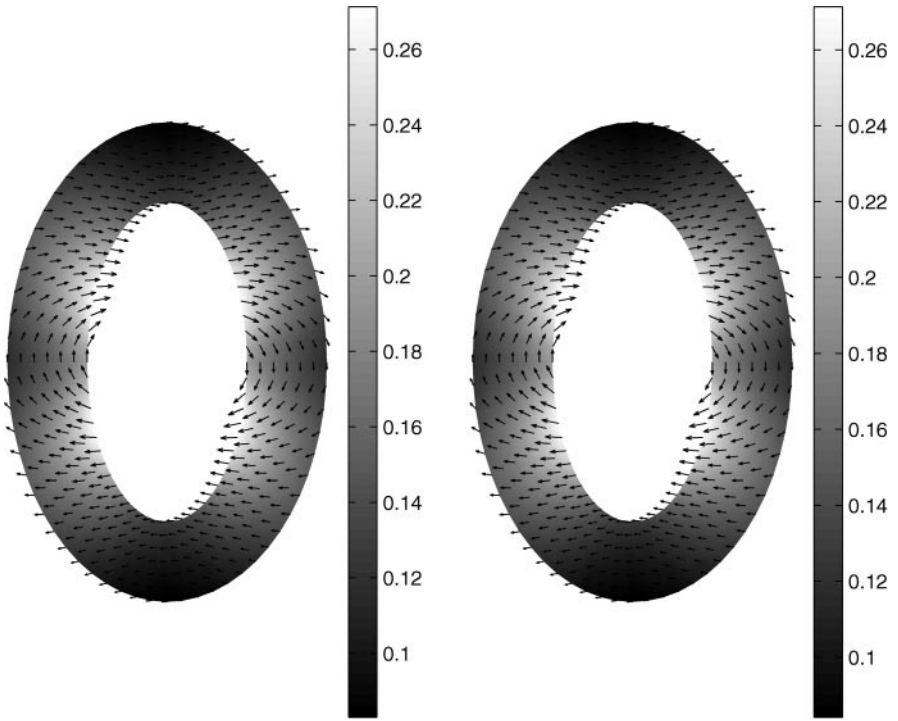


FIG. 2. Analytical (right) and computed (left) velocity fields.

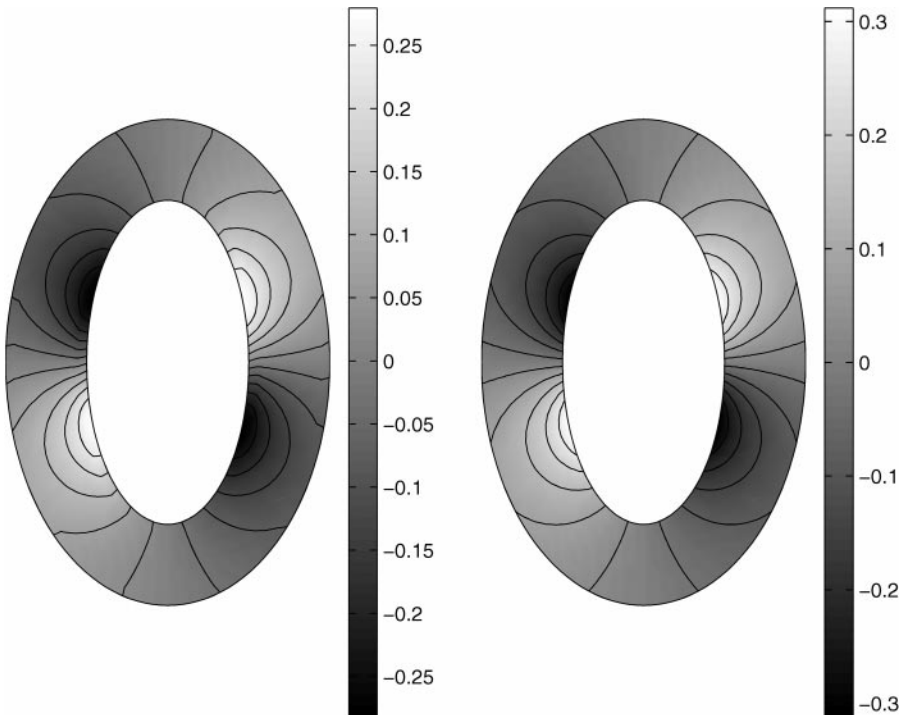
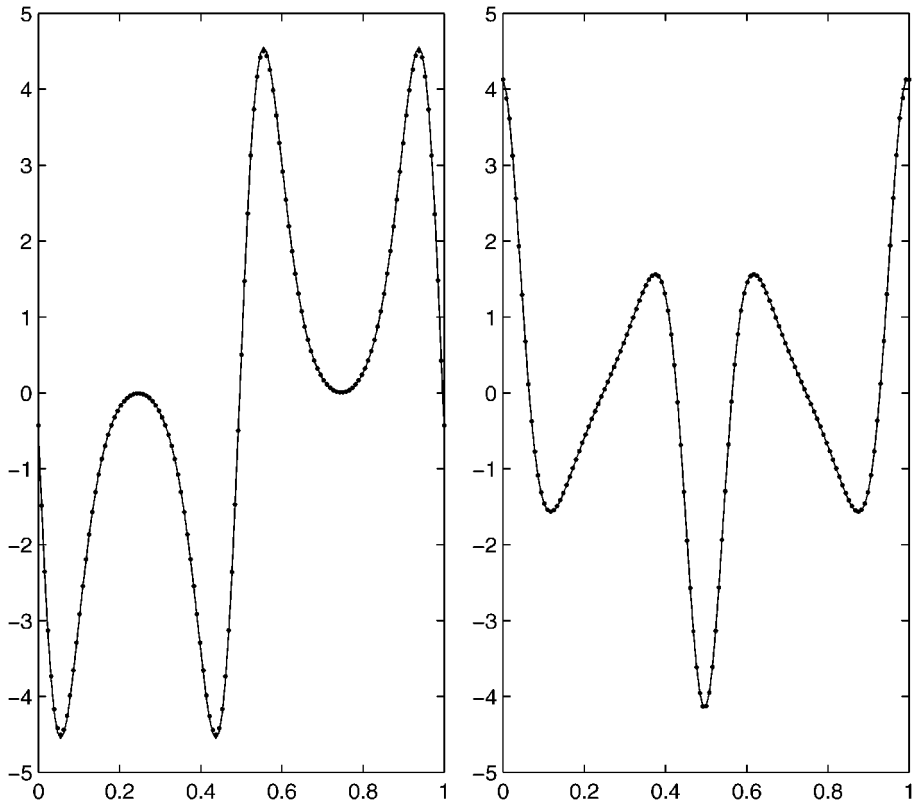


FIG. 3. Level curves of the pressure, analytical (right) and computed (left).



**FIG. 4.** First (left) and second (right) components of the stress vector. The analytical solution is represented by a line and the computed solution by dots.

The analytical (right) and computed (left) solutions are drawn (with identical scales) in Fig. 2 for the velocity vector field and in Fig. 3 for the level curves of the pressure.

The graphics of Fig. 4 compare each component of the analytical stress solution to its discrete counterpart. The exact and approximated solutions are superposed in each graphic. The analytical solution is represented by a line and the computed solution is represented by dots.

**TABLE I**  
**Convergence History and Number of Iterations**  
**for Different Values of the Mesh Parameter  $h$**

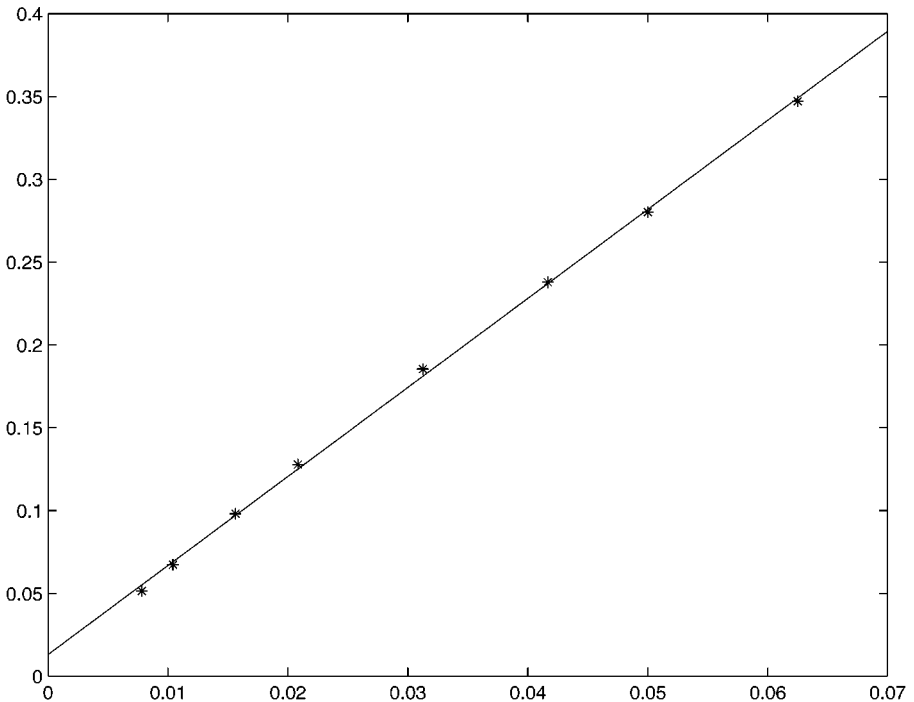
$h$	Iter.	$\ u - \bar{u}_h\ _{1,\Omega_h}$	$\ p - \bar{p}_h\ _{0,\Omega_h}$
1/8	18	0.6381	0.9251
1/16	18	0.3472	0.2490
1/32	18	0.1854	0.0925
1/64	18	0.0981	0.0426
1/128	18	0.0514	0.0214

**TABLE II**  
**Number of Iterations for Different Parameters  $\theta$**   
**and Different Auxiliary Boundaries  $\Gamma_0$**

	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$	$\theta = 0.8$	$\theta = 0.9$
$a = 2 \quad b = 3$	22	23	32	50	105
$a = 3 \quad b = 4$	—	—	33	52	109
$a = 4 \quad b = 5$	—	—	—	53	112

In Table I, we test the influence of the mesh parameter  $h$  on the error and the behavior of the iterative method. To this end, we fix the value of the relaxation parameter  $\theta$  to 0.52 and report the number of iterations and the error with respect to the exact solution in the  $H^1$ -norm for the velocity field and in the  $L^2$ -norm for the pressure. It is evident from the results reported in Table I that the convergence rate of the algorithm is independent of the number of unknowns of the problem. However, fixing the mesh parameter  $h$  to  $1/64$ , we can see in Table II that the number of iterations is sensible to the choice of  $\theta$ . We also give in Table II the number of iterations obtained by assigning different values to the minor and the major semiaxis  $a$  and  $b$  of the ellipse  $\Gamma_0$ . When no iteration number is reported the method diverges. The results show that the behavior of the iterative method is influenced by the election of the subdomain  $\Omega^-$ .

Figures 5 and 6 depict the results of Table I and show that, as expected, the error grows linearly with respect to the mesh parameter  $h$ .



**FIG. 5.** The velocity error in  $H^1$ -norm versus  $h$ .



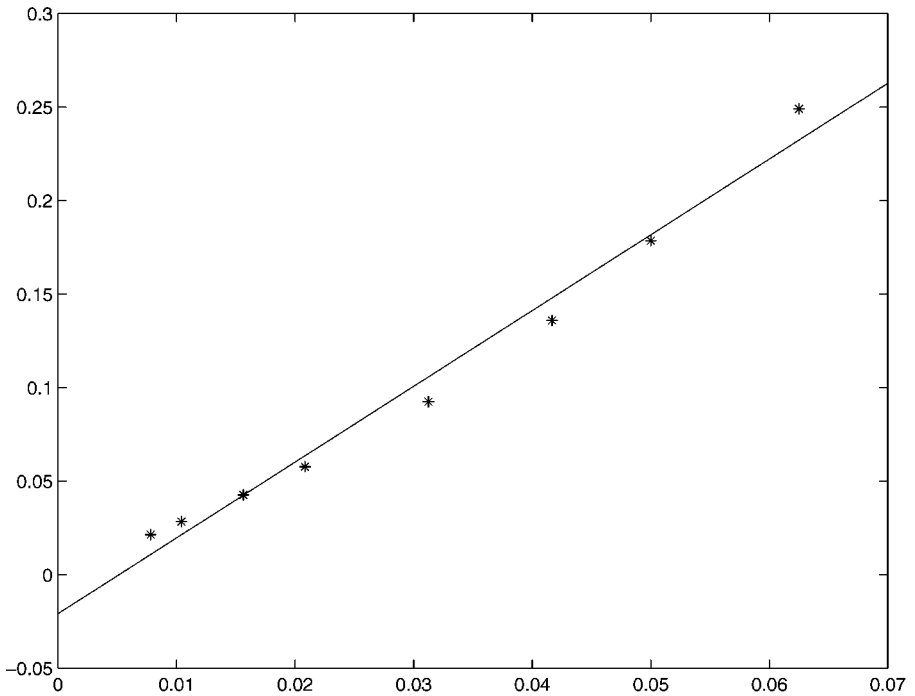


FIG. 6. The pressure error in  $L^2$ -norm versus  $h$ .

## 7. CONCLUSION AND PERSPECTIVES

We presented a method based on a coupling of boundary and finite elements, which allows one to solve numerically Stokes problems in exterior domains. This method is a simplification of the fully discrete Galerkin scheme analyzed in [21].

Instead of using exact triangulations of the computational domain (and then, curved triangles as in [21]) we showed here with numerical experiments that a discretization strategy based on polygonal approximations of the curved boundary still maintains the optimal order of convergence of the original method and matches well with the boundary element discretization. We are still able to use the elementary quadrature formulas introduced in [21] to define the fully discrete scheme. This is an important improvement since in a former implementation method proposed in [23] the singular boundary integrals are approximated with adaptative routines using Gauss 10-point and Kronrod 21-point rules, while in our case the bidimensional midpoint formula is sufficient.

We finally proposed an iterative method to efficiently solve the complicated linear system of Eq. (19). The algorithm may be viewed as a domain decomposition method (cf. [9]) which consists in solving at each iteration step an interior Neumann Stokes problem (by a mixed finite element method) and an exterior Dirichlet Stokes problem (by a boundary element method). Here again, in our opinion, this algorithm is more effective than the methodology given in [23] to deal with (19).

It is also worthwhile to mention here that, when compared with the BEM-FEM formulations based on separation of variables (see, e.g. [14] and [15]), our method allows one to use coupling interfaces of arbitrary shape. This is advantageous for problems with anisotropic geometries.

In regard to possible extensions of the method, we point out that our coupling formulation can be used verbatim for 3-D problems. However, the Galerkin scheme for the integral operator on the coupling interface is impractical in 3-D since it generates big and dense matrices that need prohibitive computational efforts to be assembled. Some fast techniques such as wavelet-type accelerations (cf. [19]) or fast multipole methods are essential to generate efficient schemes in 3-D. In a 2-D context, we think that our method may be regarded as a first step to tackle a coupling of Navier–Stokes in the interior with Stokes or Oseen in the unbounded exterior. Variational formulation for these two types of couplings have already been proposed in [6] and [7].

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